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A note on the diophantine equation $ax^m - by^n = k$

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1. INTRODUCTION

Let a, b, k, x, y be positive integers satisfying $x > 1$, $y > 1$ and $\gcd(ax, by) = 1$, and let N denote the number of integer solutions (m, n) of the equation

(1) $$ax^m - by^n = k, \quad m > 1, \ n > 1.$$

In this note we deal with the upper bound of N. In this respect, LeVeque (3) proved that if $a = b = k = 1$, then $N \leq 1$. Cao [2] proved that if $k \leq 2$, then $N \leq 4$. For the general a, b and k, Shorey [6] proved that Eq. (1) has at most nine integer solutions (m, n) satisfy $ax^m > 953k^6$. In this note we shall prove the following results:

THEOREM 1. *If* $\min(a, b) = k = 1$, *then* $N \leq 1$.

THEOREM 2. *If* $a = b = 1$ *and* $\min(x, y) \geq 10^5$, *then* $N \leq 2$.

THEOREM 3. *If* $\min(x, y) \geq e^e$, *then* $N \leq 3$.

2. LEMMAS

LEMMA 1 ([1]). *For any positive integer* r *with* $r > 2$ *and any integer* X, Y *which satisfy* $XY \neq 0$ *and* $X \neq \pm Y$, $X^r - Y^r$ *has a primitive divisor except when* $(X, Y, r) = (2, 1, 6), (2, -1, 3)$.

Let α be an algebraic number with the defining polynomial

$$a_0z^r+a_1z^{r-1}+\cdots+a_r=a_0\prod_{i=1}^{r}(z-\delta_i\alpha),\quad a_0>0,$$

where $\delta_1\alpha,\ldots,\delta_r\alpha$ are all the conjugates of α. Then

$$h(\alpha)=\frac{1}{r}(\log a_0+\sum_{i=1}^{r}\log\max(1,|\delta_i\alpha|))$$

is called Weil's height of α.

LEMMA 2 ([4]). *Let α_1,α_2 be non-zero algebraic numbers which are multiplicatively independent, and let $A_j=\max(1,h(\alpha_j)+\log 2,2e|\log\alpha_j|/R)$ $(j=1,2)$, where R is the degree of the field $\mathbb{Q}(\alpha_1,\alpha_2)$, $\log\alpha_j$ is any non-zero determination of the logarithm of α_j. If $\Lambda=b_1\log\alpha_1-b_2\log\alpha_2\neq 0$ for some positive integers b_1,b_2, then*

$$|\Lambda|>\exp(-500A_1A_2R^4(7.5+\log B)^2),$$

where $B=\max(b_1,b_2)$.

LEMMA 3 ([5]). *Let $\alpha_1,\ldots,\alpha_n$ be non-zero algebraic numbers with heights $H_1,\ldots,H_n$ which are multiplicatively independent, and let $A_i'=\max(e^e,H_i)$ $(i=1,\ldots,n)$. If $A_i'\leq\cdots\leq A_n'$ and $\Lambda'=b_1'\log\alpha_1+\cdots+b_n'\log\alpha_n\neq 0$ for some integers $b_1',\ldots,b_n'$, then*

$$|\Lambda'|>\exp(-2^{61n+47}n^{10n}d^{10n+10}(\log B')(\prod_{i=1}^{n}\log A_i')(\log\prod_{j=1}^{n-1}\log A_j')),$$

where d is the degree of the field $\mathbb{Q}(\alpha_1,\ldots,\alpha_n)$, $B'=\max(e^e,|b_1'|,\ldots,|b_n'|)$.

3. PROOFS

PROOF OF THEOREM 1. We suppose that Eq. (1) has two integer solutions (m_i,n_i) $(i=1,2)$ with $m_1<m_2$. If $a=1$, then

(2) $\quad x^{m_1}-1=by^{n_1},\quad m_1>1,\ n_1>1$

and

(3) $\quad x^{m_2}-1=by^{n_2},\quad m_2>m_1,\ n_2>n_1.$

By Lemma 1, we get from (2) and (3) that $x=2$, $m_2=6$ and $by^{n_2}=63=r\cdot 3^2$. It is impossible since $n_2\geq 3$. Hence $N\leq 1$.

If $b=1$, then

(4) $\quad y^{n_1}+1=ax^{m_1},\quad n_1>1,\ m_1>1,$

and

(5) $\quad y^{n_2}+1=ax^{m_2},\quad n_2>n_1,\ m_2>m_1.$

Since $n_2 > n_1$, we have $n_2 = \alpha n_1 + \beta$, where α, β are integers satisfy $\alpha > 0$ and $0 \le \beta < n_1$. From (4) and (5), we get $-1 \equiv y^{n_2} = y^{\alpha n_1 + \beta} \equiv (-1)^{\alpha} y^{\beta} \pmod{ax^{m_1}}$. It implies that $\beta = 0$ and $2 \nmid \alpha$. Hence, by (4) and (5),

$$(6) \qquad \begin{cases} x^{m_2 - m_1} = \dfrac{(y^{n_1})^{\alpha} + 1}{y^{n_1} + 1} = (ax^{m_1})^{\alpha - 1} - \dbinom{\alpha}{\alpha - 1}(ax^{m_1})^{\alpha - 2} \\ \qquad\qquad + \cdots + (-1)^{\alpha - 2}\dbinom{\alpha}{2} ax^{m_1} + (-1)^{\alpha - 1}\dbinom{\alpha}{1}. \end{cases}$$

Notice that $2 \nmid \alpha$ and $m_2 > m_1$. We see from (6) that $2 \nmid x$ and $\alpha \equiv 0 \pmod{x}$. If $x^{\gamma} \,\|\, \alpha$, then

$$\binom{\alpha}{i}(ax^{m_1})^{i-1} = \alpha \binom{\alpha - 1}{i - 1} \frac{(ax^{m_1})^{i-1}}{i} \equiv 0 \pmod{x^{\gamma + 1}} \quad \text{for } i > 1.$$

Therefore, we obtain from (6) that

$$(7) \qquad \alpha \equiv 0 \pmod{x^{m_2 - m_1}}.$$

Since $m_1 > 1$ and $n_1 > 1$, (6) is impossible by (7). Thus $N \le 1$. The theorem is proved.

PROOF OF THEOREM 2. If $N > 2$, then Eq. (1) has three integer solutions (m_i, n_i) $(i = 1, 2, 3)$ with $m_1 < m_2 < m_3$. Notice that $\gcd(x, y) = 1$. Let s, t be the least positive integers satisfy

$$(8) \qquad x^s \equiv 1 \pmod{y^{n_1}}, \quad y^t \equiv 1 \pmod{x^{m_1}}.$$

Since

$$(9) \qquad x^{m_i} - y^{n_i} = k, \quad i = 1, 2, 3,$$

we get from

$$(10) \qquad x^{m_i} \equiv k \pmod{y^{n_i}}, \quad i = 1, 2, 3,$$

that

$$(11) \qquad x^{m_{j+1} - m_j} \equiv 1 \pmod{y^{n_j}}, \quad j = 1, 2.$$

By (8) and (11), $m_2 - m_1 = su_1$, where $u_1 \in \mathbb{N}$. If $\gcd(y^{n_2 - n_1}, (x^{m_2 - m_1} - 1)/y^{n_1}) = d > 1$, then we get $x^{m_1} \equiv k \pmod{y^{n_1} d}$ by (10). This would contradict (9). Therefore, $\gcd(y, (x^s - 1)/y^{n_1}) = 1$, and by (11),

$$(12) \qquad m_{j+1} - m_j = su_j y^{n_j - n_1}, \quad u_j \in \mathbb{N},\ j = 1, 2.$$

Similarly, from

$$(13) \qquad y^{n_i} \equiv -k \pmod{x^{m_i}}, \quad i = 1, 2, 3,$$

and

$$(14) \qquad y^{n_{j+1} - n_j} \equiv 1 \pmod{x^{m_j}}, \quad j = 1, 2,$$

we obtain

(15) $\quad n_{j+1}-n_j=tv_jx^{m_j-m_1}, \quad v_j\in\mathbb{N},\ j=1,2$

by (8).

From (8), (9) and (15),

$$y^{n_3}>y^{n_3-n_2}\geq y^{tx^{m_2-m_1}}>x^{m_1x^{m_2-m_1}}>k^{x^{m_2-m_1}}.$$

We get from (9) that

$$(16)\qquad \begin{cases} 0<\Lambda=m_3\log x-n_3\log y=\dfrac{2k}{2y^{n_3}+k}\displaystyle\sum_{l=0}^{\infty}\frac{1}{2l+1}\left(\frac{k}{2y^{n_3}+k}\right)^{2l} \\ <\dfrac{4k}{2y^{n_3}+k}=\dfrac{4k}{x^{m_3}+y^{n_3}}. \end{cases}$$

Since $\gcd(x,y)=1$, by Lemma 2, we have

(17) $\quad \Lambda>\exp(-500(2e\log x)(2e\log y)(7.5+\log\max(m_3,n_3))^2).$

On combining (17) with (16), we obtain

$$\log 4k+14\,800(\log x)(\log y)(7.5+\log\max(m_3,n_3))^2>\begin{cases} m_3\log x, & \text{if } m_3\geq n_3,\\ n_3\log y, & \text{if } m_3<n_3,\end{cases}$$

whence we conclude that

$$(18)\qquad \begin{cases} m_3<4\cdot 10^7(\log y)(\log\log y)^2, & \text{if } m_3\geq n_3,\\ n_3<4\cdot 10^7(\log x)(\log\log x)^2, & \text{if } m_3<n_3.\end{cases}$$

If $m_3\geq n_3$, from (12) and (18), we get

(19) $\quad 17.51+\log\log y+2\log\log\log y>\log m_3>\log(m_3-m_2)\geq\log y^{n_2-n_1}.$

Since $x^{m_1}>y^{n_1}$, we find from (14) that $n_2-n_1\geq 2$. Hence, we obtain from (19) that $y<10^5$.

If $m_3<n_3$, then from (15) and (18) we get

$$17.51+\log\log x+2\log\log\log x>\log n_3>\log(n_3-n_2)\geq\log x^{m_2-m_1},$$

whence we conclude that

$$(20)\qquad x<\begin{cases} 10^{10}, & \text{if } m_2-m_1=1,\\ 10^5, & \text{otherwise.}\end{cases}$$

Further, if $m_2-m_1=1$, then $x-1\geq y^{n_1}\geq y^2$ by (8) and (12). This implies that $y<10^5$ by (20). Thus $N\leq 2$ for $\min(x,y)\geq 10^5$. The theorem is proved.

PROOF OF THEOREM 3. If $N>3$, then Eq. (1) has four integer solutions (m_i,n_i) $(i=1,\ldots,4)$ with $m_1<\cdots<m_4$. Since $\gcd(ax,by)=1$, by much the same argument as in the proof of Theorem 2, we have

(21) $\quad x^{m_{j+1}-m_j}\equiv 1 \pmod{by^{n_j}}, \quad y^{n_{j+1}-n_j}\equiv 1 \pmod{ax^{m_j}}, \quad j=1,2,3,$

and

$$(22)\qquad m_{j+1}-m_j=s'u_j'y^{n_j-n_1},\quad n_{j+1}-n_j=t'v_j'x^{m_j-m_1},\quad u_j', v_j'\in\mathbb{N},\ j=1,2,3,$$

where s', t' are positive integers satisfying

$$(23)\qquad x^{s'}\equiv 1 \pmod{by^{n_1}},\quad y^{t'}\equiv 1 \pmod{ax^{m_1}}.$$

By (21) and (22),

$$(24)\qquad \log ax^{m_4}>\log by^{n_4}>n_4>n_4-n_3>x^{m_3-m_2}\geq x^{y^{n_2-n_1}}>x^{ax^{m_1}}>x^k.$$

So we have

$$(25)\qquad 0<\Lambda'=\log\frac{a}{b}+m_4\log x-n_4\log y<\frac{4k}{ax^{m_4}+by^{n_4}},$$

since $ax^{m_4}-by^{n_4}=k$.

Notice that $\gcd(ax, by)=1$ and $\min(x, y)\geq e^e$. By Lemma 3, we have

$$\Lambda'>\exp(-2^{230}3^{30}(\log\max(m_4, n_4))$$
$$\times(\log\max(e^e, a, b)(\log x)(\log y)(\log\log x+\log\log y)).$$

On combining this with (25),

$$(26)\qquad \begin{cases}\log 4k+2^{230}3^{30}(\log\max(m_4, n_4))(\log\max(e^e, a, b))(\log x)(\log y)\\ \quad(\log\log x+\log\log y)>\log(ax^{m_4}+by^{n_4}).\end{cases}$$

We get from (24) that

$$(27)\qquad \log\log\max(e^e, a, b)+\log 4\max(m_4, n_4)>\log 4k.$$

When $m_4\geq n_4$, by (26) and (27)

$$(28)\qquad \begin{cases}\log\log\max(e^e, a, b)+(4+2^{230}3^{30}(\log\max(e^e, a, b))(\log x)(\log y)\\ \quad(\log\log x+\log\log y))\log m_4>m_4\log x.\end{cases}$$

Put

$$(29)\qquad \begin{cases}m_4=2^{230}3^{30}c(\log\max(e^e, a, b))(\log\log\max(e^e, a, b))(\log y)\\ \qquad\times(\log\log x+\log\log y)^2.\end{cases}$$

Substituting (29) into (28), we conclude that $c<200$. Hence, by (29),

$$(30)\qquad \begin{cases}m_4<10^{86}(\log\max(e^e, a, b))(\log\log\max(e^e, a, b))(\log y)\\ \qquad\times(\log\log x+\log\log y)^2.\end{cases}$$

By (21), (22) and (23),

$$m_4>m_4-m_3\geq s'y^{n_3-n_1}=s'y^{(n_3-n_2)+(n_2-n_1)}>s'ax^{m_1}y^{t'x^{m_2-m_1}}>s'ax^{m_1}y^{t'by^{n_1}}.$$

On combining this with (30), we get

$$(31)\qquad \begin{cases}198+\log\log\max(e^e, a, b)+\log\log\log\max(e^e, a, b)+\log\log y\\ \qquad+2\log(\log\log x+\log\log y)>\log s'+\log a+m_1\log x+t'by^{n_1}\log y.\end{cases}$$

From (23),

$$\log s' > \log n_1 + \log\log y - \log\log x \geq \log 2 + \log\log y - \log\log x.$$

Substituting this into (31),

$$(31)\qquad \begin{cases} 198 + \log\log\max(e^e, a, b) + \log\log\log\max(e^e, a, b) + \log\log x \\ \quad + 2\log(\log\log x + \log\log y) > \log a + m_1 \log x + t'by^{n_1} \log y. \end{cases}$$

Further, by (23), $t' \geq n_1 + 1 \geq 3$. Since

$$(33)\qquad \log x + \log y > \log\log x + \log\log y + 2\log(\log\log x + \log\log y),$$

we get from (32) that

$$198 + \log\log\max(e^e, a, b) + \log\log\log\max(e^e, a, b)$$
$$> \log a + (m_1 - 1)\log x + (3by^{n_1} - 1)\log y > \log a + 1950b,$$

since $\min(x, y) \geq e^e$. This is a contradiction. Thus $N \leq 3$.

When $m_4 < n_4$. By much the same argument as above, we obtain from (26) and (27) that

$$(34)\qquad \begin{cases} n_4 < 10^{86}(\log\max(e^e, a, b))(\log\log\max(e^e, a, b))(\log x) \\ \qquad \times (\log\log x + \log\log y)^2. \end{cases}$$

By (21), (22) and (23),

$$n_4 > n_4 - n_3 \geq t'x^{m_3 - m_1} = t'x^{(m_3 - m_2) + (m_2 - m_1)} > t'by^{n_1}x^{s'y^{n_2 - n_1}} > t'by^{n_1}x^{s'ax^{m_1}}.$$

On combining this with (34),

$$(35)\qquad \begin{cases} 198 + \log\log\max(e^e, a, b) + \log\log\log\max(e^e, a, b) + \log\log x \\ \quad + 2\log(\log\log x + \log\log y) > \log t' + \log b + n_1 \log y + s'ax^{m_1} \log x. \end{cases}$$

From (23),

$$\log t' > \log m_1 + \log\log x - \log\log y \geq \log 2 + \log\log x - \log\log y.$$

Substituting this into (35), we obtain

$$198 + \log\log\max(e^e, a, b) + \log\log\log\max(e^e, a, b)$$
$$> \log b + (n_1 - 1)\log y + (s'ax^{m_1} - 1)\log x$$
$$\geq \log b + \log y + (ax^{m_1} - 1)\log x > \log b + 650a$$

by (33), which is a contradiction. The proof is complete.

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